Kenneth Konyndyk, *Introductory Modal Logic* (University of Notre Dame Press, 1986)
P. 10, 1.5 Excersises 2

(a) $(\neg p \supset \neg q) \supset ((\neg p \supset q) \supset p)$

1	$\neg p \supset \neg q$	asp
2	$\neg p \supset q$	asp
3	$\neg p$	asp
4	$\neg p \supset \neg q$	reit
5	$\neg q$	MP
6	$\neg p \supset q$	reit
7	q	MP
8	$\bot$	
9	p	RAA
10	$(\neg p \supset q) \supset p$	impl intro
11	$(\neg p \supset \neg q) \supset ((\neg p \supset q) \supset p)$	impl intro

(b) $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$

1	$p \supset (q \supset r)$	asp
2	$p \supset q$	asp
3	p	asp
4	$p \supset q$	reit
5	q	MP
6	$p \supset (q \supset r)$	reit
7	$q \supset r$	MP
8	r	MP
9	$p \supset r$	impl intro
10	$(p \supset q) \supset (p \supset r)$	impl intro
11	$(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$	impl intro

(c) $(p \equiv (q \supset r)) \equiv (((p \& q) \supset r) \& ((q \supset r) \supset p)$

1	$\vert\ \underline{p \equiv (q \supset r)}$	asp
2	$\vert\ \vert\ \underline{p\&q}$	asp
3	$\vert\ \vert\ p$	conj (&) elim
4	$\vert\ \vert\ p \equiv (q \supset r)$	reit
5	$\vert\ \vert\ q \supset r$	$\equiv$ elim
6	$\vert\ \vert\ q$	conj elim
7	$\vert\ \vert\ r$	impl ($\supset$) elim
8	$\vert\ (p\&q) \supset r$	impl intro
9	$\vert\ \vert\ \underline{q \supset r}$	asp
10	$\vert\ \vert\ p \equiv (q \supset r)$	reit
11	$\vert\ \vert\ p$	$\equiv$ elim
12	$\vert\ (q \supset r) \supset p$	impl intro
13	$\vert\ ((p\&q) \supset r)\&((q \supset r) \supset p)$	conj intro
14	$(p \equiv (q \supset r)) \supset (((p\&q) \supset r)\&((q \supset r) \supset p))$	impl intro
15	$\vert\ \underline{((p\&q) \supset r)\&((q \supset r) \supset p)}$	asp
16	$\vert\ \vert\ \underline{p}$	asp
17	$\vert\ \vert\ \vert\ \underline{q}$	asp
18	$\vert\ \vert\ \vert\ p$	reit
19	$\vert\ \vert\ \vert\ p\&q$	conj intro
20	$\vert\ \vert\ \vert\ ((p\&q) \supset r)\&((q \supset r) \supset p))$	reit
21	$\vert\ \vert\ \vert\ (p\&q) \supset r$	conj elim
22	$\vert\ \vert\ \vert\ r$	impl elim
23	$\vert\ \vert\ q \supset r$	impl intro
24	$\vert\ p \supset (q \supset r)$	impl intro
25	$\vert\ \vert\ \underline{q \supset r}$	asp
26	$\vert\ \vert\ ((p\&q) \supset r)\&((q \supset r) \supset p)$	reit
27	$\vert\ \vert\ (q \supset r) \supset p$	conj elim
28	$\vert\ \vert\ p$	impl elim
29	$\vert\ (q \supset r) \supset p$	impl intro
30	$\vert\ p \equiv (q \supset r)$	$\equiv$ intro
31	$(((p\&q) \supset r)\&((q \supset r) \supset p)) \supset (p \equiv (q \supset r))$	impl intro
32	$(p \equiv (q \supset r)) \equiv (((p\&q) \supset r)\&((q \supset r) \supset p))$	$\equiv$ intro

(d) $p \supset (q \vee \neg q)$

1	$\mid\ \underline{p}$	asp
2	$\mid\ \mid\ \underline{\neg(q \vee \neg q)}$	asp
3	$\mid\ \mid\ \mid\ \underline{q}$	asp
4	$\mid\ \mid\ \mid\ q \vee \neg q$	disj intro
5	$\mid\ \mid\ \mid\ \bot$	
6	$\mid\ \mid\ \neg q$	RAA
7	$\mid\ \mid\ q \vee \neg q$	disj intro
8	$\mid\ \neg\neg(q \vee \neg q)$	RAA
9	$\mid\ q \vee \neg q$	DN
10	$p \supset (q \vee \neg q)$	impl intro

(d-2) $p \supset (q \vee \neg q)$

1	$\mid\ \underline{p}$	asp
2	$\mid\ \mid\ \underline{\neg(q \vee \neg q)}$	asp
3	$\mid\ \mid\ \neg q \& \neg\neg q$	De Morgan's Theorem
4	$\mid\ \mid\ \neg q$	conj elim (& elim)
5	$\mid\ \mid\ \neg\neg q$	conj elim
6	$\mid\ \mid\ \bot$	
7	$\mid\ \neg\neg(q \vee \neg q)$	RAA
8	$\mid\ q \vee \neg q$	DN
9	$p \supset (q \vee \neg q)$	impl intro

P. 38, 2.3.c A

1. $p \rightarrow q, q \rightarrow r \therefore p \rightarrow r$

1	$p \rightarrow q$	prem
2	$q \rightarrow r$	prem
3	$\Box(p \supset q)$	def
4	$\Box(q \supset r)$	def
5	$\quad \Box$	
6	$\quad p \supset q$	T-reit
7	$\quad q \supset r$	T-reit
8	$\quad\quad p$	asp
9	$\quad\quad p \supset q$	reit
10	$\quad\quad q$	MP
11	$\quad\quad q \supset r$	reit
12	$\quad\quad r$	MP
13	$\quad p \supset r$	impl intro
14	$\Box(p \supset r)$	nec intro
15	$p \rightarrow r$	def

2. $p \to q, \Box p, q \to r \therefore \Box r$

1	$p \to q$	prem
2	$\Box p$	prem
3	$q \to r$	prem
4	$\Box(p \supset q)$	def
5	$\Box(q \to r)$	def
6	$\Box$	
7	p	T-reit
8	$p \supset q$	T-reit
9	q	MP
10	$q \supset r$	T-reit
11	r	MP
12	$\Box r$	nec intro

3. $\Box(p \& q) \therefore \Box q$

1	$\Box(p \& q)$	prem
2	$\Box$	
3	$p \& q$	T-reit
4	q	conj elim
5	$\Box q$	nec intro

4. $\Box(p \vee q), \Box\neg p \therefore \Box q$

1	$\Box(p \vee q)$	prem
2	$\Box\neg p$	prem
3	$\Box$	
4	$p \vee q$	T-reit
5	$\neg p$	T-reit
6	q	disj elim
7	$\Box q$	nec intro

5. $\Box(p \vee q), p \rightarrow q \therefore \Box q$

1	$\Box(p \vee q)$	prem
2	$p \rightarrow q$	prem
3	$\Box(p \supset q)$	def
4	$\quad \Box$	
5	$\quad p \vee q$	T-reit
6	$\quad p \supset q$	T-reit
7	$\quad\quad q$	asp
8	$\quad\quad q$	rep
9	$\quad q \supset q$	impl intro
10	$\quad q$	disj elim
11	$\Box q$	nec intro

6. $p \to q, p \to \Box r, q \to \Box\neg r \therefore \Box\neg p$

1	$p \to q$	prem
2	$p \to \Box r$	prem
3	$q \to \Box\neg r$	prem
4	$\Box(p \supset q)$	def
5	$\Box(p \supset \Box r)$	def
6	$\Box(q \supset \Box\neg r)$	def
7	$\Box$	
8	$p \supset q$	T-reit
9	$p \supset \Box r$	T-reit
10	$q \supset \Box\neg r$	Treit
11	p	asp
12	$p \supset q$	reit
13	q	MP
14	$p \supset \Box r$	reit
15	$\Box r$	MP
16	$q \supset \Box\neg r$	reit
17	$\Box\neg r$	MP
18	r	nec elim
19	$\neg r$	nec elim
20	$\bot$	
21	$\neg p$	RAA
22	$\Box\neg p$	nec intro

P. 39, 2.3.c B

1. $p \to p$

1	$\Box$	
2	p	asp
3	p	rep
4	$p \supset p$	impl intro
5	$\Box(p \supset p)$	nec intro
6	$p \to p$	def

2. $\Box\Box p \to \Box p$

1	$\Box$	
2	$\Box\Box p$	asp
3	$\Box p$	nec elim
4	$\Box\Box p \supset \Box p$	impl intro
5	$\Box(\Box\Box p \supset \Box p)$	nec intro
6	$\Box\Box p \to \Box p$	def

3. $(p \to q) \to (\Box p \supset \Box q)$

1	$\Box$	
2	$p \to q$	asp
3	$\Box(p \supset q)$	def
4	$\Box p$	asp
5	$\Box(p \supset q)$	reit
6	$\Box$	
7	p	T-reit
8	$p \supset q$	T-reit
9	q	MP
10	$\Box q$	nec intro
11	$\Box p \supset \Box q$	impl intro
12	$(p \to q) \supset (\Box p \supset \Box q)$	impl intro
13	$\Box((p \to q) \supset (\Box p \supset \Box q))$	nec intro
14	$(p \to q) \to (\Box p \supset \Box q)$	def

5. $\Box p \to (q \to p)$

1	$\Box$	
2	$\Box p$	asp
3	$\Box$	
4	q	asp
5	p	T-reit
6	$q \supset p$	impl intro
7	$\Box(q \supset p)$	nec intro
8	$\Box p \supset \Box(q \supset p)$	impl intro
9	$\Box(\Box p \supset \Box(q \supset p))$	nec intro
10	$\Box p \to (q \to p)$	def

6. $\Box\neg p \to (p \to q)$

1	$\Box$	
2	$\Box\neg p$	asp
3	$\Box$	
4	p	asp
5	$\neg q$	asp
6	$\neg p$	T-reit
7	p	reit
8	$\bot$	
9	q	RAA
10	$p \supset q$	impl intro
11	$\Box(p \supset q)$	nec intro
12	$\Box\neg p \supset \Box(p \supset q)$	impl intro
13	$\Box(\Box\neg p \supset \Box(p \supset q))$	nec intro
14	$\Box\neg p \to (p \to q)$	def

7. $(p \to (q \& \neg q)) \to \Box\neg p$

1	$\Box$	
2	$\Box(p \supset (q \& \neg q))$	asp
3	$\Box$	
4	p	asp
5	$p \supset (q \& \neg q)$	T-reit
6	$q \& \neg q$	MP
7	q	& elim
8	$\neg q$	& elim
9	$\bot$	
10	$\neg p$	RAA
11	$\Box\neg p$	nec intro
12	$\Box(p \supset (q \& \neg q)) \supset \Box\neg p)$	impl intro
13	$\Box(\Box(p \supset (q \& \neg q)) \supset \Box\neg p))$	impl intro
14	$(p \to (q \& \neg q)) \to \Box\neg p$	def

8. $((p \to q) \& (\neg p \to q)) \to \Box q$

1	$\Box$	
2	$\Box(p \supset q) \& \Box(\neg p \supset q)$	asp
3	$\Box(p \supset q)$	conj (&) elim
4	$\Box(\neg p \supset q)$	conj elim
5	$\Box$	
6	$p \supset q$	T-reit
7	$\neg p \supset q$	T-reit
8	$\neg q$	asp
9	$\neg p$	MT (modus tollens)
10	q	MP (modus ponens)
11	$\perp$	
12	q	RAA
13	$\Box q$	nec intro
14	$(\Box(p \supset q) \& \Box(\neg p \supset q)) \supset \Box q$	impl intro
15	$\Box((\Box(p \supset q) \& \Box(\neg p \supset q)) \supset \Box q)$	nec intro
16	$((p \to q) \& (\neg p \to q)) \to \Box q$	def

P. 44, 2.3.g.A.

1. $\Box p \to \Box(p \vee q)$

1	$\Box$	
2	$\Box p$	asp
3	$\Box$	
4	p	T-reit
5	$p \vee q$	disj intro
6	$\Box(p \vee q)$	nec intro
7	$\Box p \supset \Box(p \vee q)$	impl intro
8	$\Box(\Box p \supset \Box(p \vee q))$	nec intro
9	$\Box p \to \Box(p \vee q)$	def

2. $(p \to q) \supset (\neg q \to \neg p)$

1	$p \to q$	asp
2	$\Box(p \supset q)$	def
3	$\Box$	
4	$\neg q$	asp
5	$p \supset q$	T-reit
6	$\neg p$	MT
7	$\neg q \supset \neg p$	impl intro
8	$\Box(\neg q \supset \neg p)$	nec intro
9	$\neg q \to \neg p$	def
10	$(p \to q) \supset (\neg q \to \neg p)$	impl intro

3. $\neg\Diamond p \supset \neg\Box p$

1	$\neg\Diamond p$	asp
2	$\Box\neg p$	def
3	$\neg p$	nec elim
4	$\Diamond\neg p$	poss intro
5	$\neg\Box p$	def
6	$\neg\Diamond p \supset \neg\Box p$	

4. $(p \to r)\&(q \to s), \Box p \& \Box q \therefore \Box r \vee \Box s$

1	$(p \to r)\&(q \to s)$	prem
2	$\Box p \& \Box q$	prem
3	$\Box(p \supset r)\&\Box(q \supset s)$	def
4	$\Box(p \supset r)$	& elim
5	$\Box p$	& elim
6	$\quad \Box$	
7	$\quad p \supset r$	T-reit
8	$\quad p$	T-reit
9	$\quad r$	MP
10	$\Box r$	nec intro
11	$\Box r \vee \Box s$	disj intro

5. $(p \to r) \vee (q \to r), \Box(p \& q) \therefore \Box r$

1	$(p \to r) \vee (q \to r)$	prem
2	$\Box(p \& q)$	prem
3	$\Box(p \supset r) \vee \Box(q \supset r)$	def
4	$\Box(p \supset r)$	asp
5	$\Box$	
6	$p \& q$	T-reit
7	p	conj elim
8	$p \supset r$	T-reit
9	r	MP
10	$\Box r$	nec intro
11	$\Box(p \supset r) \supset \Box r$	impl intro
12	$\Box(q \supset r)$	asp
13	$\Box$	
14	$p \& q$	T-reit
15	q	conj elim
16	$q \supset r$	T-reit
17	r	MP
18	$\Box r$	nec intro
19	$\Box(q \supset r) \supset \Box r$	impl intro
20	$\Box r$	disj elim

P. 45, 2.3.g C

1. $\Diamond p \supset \Diamond(p \vee q)$

1	$\Diamond p$	prem
2	$\Box$	
3	p	asp
4	$p \vee q$	disj intro
5	$p \supset (p \vee q)$	impl intro
6	$\Box(p \supset (p \vee q))$	nec intro
7	$\Diamond(p \vee q)$	poss elim
8	$\Diamond p \supset \Diamond(p \vee q)$	impl intro

2. $(\Box p \& \Box q) \supset \Box(p \& q)$

1	$(\Box p \& \Box q)$	prem
2	$\Box p$	conj elim
3	$\Box q$	conj elim
4	$\Box$	
5	p	T-reit
6	q	T-reit
7	$p \& q$	conj intro
8	$\Box(p \& q)$	nec intro
9	$(\Box p \& \Box q) \supset \Box(p \& q)$	impl intro

3. $\Box(p\&q) \supset (\Box p\&\Box q)$

1	$\Box(p\&q)$	prem
2	$\quad\Box$	
3	$\quad p\&q$	T-reit
4	$\quad p$	conj elim
5	$\quad q$	conj elim
6	$\Box p$	nec intro
7	$\Box q$	nec intro
8	$\Box p\&\Box q$	conj intro
9	$\Box(p\&q) \supset (\Box p\&\Box q)$	impl intro

4. $(\Box p \vee \Box q) \supset \Box(p \vee q)$

1	$\Box p \vee \Box q$	prem
2	$\quad\Box p$	asp
3	$\quad\quad\Box$	
4	$\quad\quad p$	T-reit
5	$\quad\quad p \vee q$	disj intro
6	$\quad\Box(p \vee q)$	nec intro
7	$\quad\Box q$	asp
8	$\quad\quad\Box$	
9	$\quad\quad q$	T-reit
10	$\quad\quad p \vee q$	disj intro
11	$\quad\Box(p \vee q)$	nec intro
12	$\Box(p \vee q)$	disj elim
13	$(\Box p \vee \Box q) \supset \Box(p \vee q)$	impl intro

5. $(\Diamond p \& \neg\Diamond q) \supset \Diamond(p \vee q)$

1	$\Diamond p \& \neg\Diamond q$	prem
2	$\Diamond p$	conj elim
3	$\Box$	
4	p	asp
5	$p \vee q$	disj intro
6	$p \supset (p \vee q)$	impl intro
7	$\Box(p \supset (p \vee q))$	nec intro
8	$\Diamond(p \vee q)$	poss elim
9	$(\Diamond p \& \neg\Diamond q) \supset \Diamond(p \vee q)$	impl intro

P. 50, 2.4.b. B

1. $\Diamond\Box\Diamond p \supset \Diamond p$

1	$\Diamond\Box\Diamond p$	asp
2	$\Box$	
3	$\Box\Diamond p$	asp
4	$\Diamond p$	nec elim
5	$\Box\Diamond p \supset \Diamond p$	impl intro
6	$\Box(\Box\Diamond p \supset \Diamond p)$	nec intro
7	$\Diamond\Diamond p$	poss elim
8	$\neg\Diamond p$	asp
9	$\Box\neg p$	def
10	$\Box\Box\neg p$	AS4
11	$\Box\neg\Diamond p$	def
12	$\neg\Diamond\Diamond p$	def
13	$\Diamond\Diamond p$	reit
14	$\perp$	
15	$\Diamond p$	RAA
16	$\Diamond\Box\Diamond p \supset \Diamond p$	impl intro

2. $(p \to q) \supset (r \to (p \to q))$

1	$\Box(p \supset q)$	asp
2	$\Box\Box(p \supset q)$	AS4
3	$\Box$	
4	r	asp
5	$\Box(p \supset q)$	T-reit
6	$r \supset \Box(p \supset q)$	impl intro
7	$\Box(r \supset \Box(p \supset q))$	nec intro
8	$\Box(p \supset q) \supset \Box(r \supset \Box(p \supset q))$	impl intro
9	$(p \to q) \supset (r \to (p \to q))$	def

3. $\Box p \supset (q \to \Box p)$

1	$\Box p$	asp
2	$\Box$	
3	q	asp
4	$\Box p$	S4-reit
5	$q \supset \Box p$	imol intro
6	$\Box(q \supset \Box p)$	nec intro
7	$\Box p \supset \Box(q \supset \Box p)$	impl intro
8	$\Box p \supset (q \to \Box p)$	def

4. $(p \supset \Box q) \supset (p \supset (r \to \Box q))$

1	$p \supset \Box p$	asp
2	$\quad p$	asp
3	$\quad p \supset \Box p$	reit
4	$\quad \Box p$	MP
5	$\quad \Box\Box p$	AS4
6	$\quad\quad \Box$	
7	$\quad\quad\quad r$	asp
8	$\quad\quad\quad \Box p$	T-reit
9	$\quad\quad r \supset \Box p$	impl intro
10	$\quad \Box(r \supset \Box p)$	nec intro
11	$p \supset \Box(r \supset \Box p)$	impl intro
12	$p \supset (r \to \Box p)$	def
13	$(p \supset \Box q) \supset (p \supset (r \to \Box q))$	impl intro

P. 53, 2.5.b

1. $\Box(p \vee \Box q) \supset (\Box p \vee \Box q)$

1	$\Box(p \vee \Box q)$	asp
2	$\neg(\Box p \vee \Box q)$	asp
3	$\neg\Box p \& \neg\Box q$	De Morgan's Theorem
4	$\Diamond\neg p \& \Diamond\neg q$	def
5	$\Diamond\neg p$	conj elim
6	$\Diamond\neg q$	conj elim
7	$\Box$	
8	$p \vee \Box q$	T-reit
9	$\Diamond\neg q$	S5-reit
10	$\neg\Box q$	def
11	p	disj elim
12	$\Box p$	nec intro
13	$\neg\Box p$	def
14	$\bot$	
15	$\Box p \vee \Box q$	RAA
16	$\Box(p \vee \Box q) \supset (\Box p \vee \Box q)$	impl intro

2. $\Box(p \vee q) \supset (\Box p \vee \Diamond q)$

1	$\mid\ \Box(p \vee q)$	
2	$\mid\ \mid\ \neg(\Box p \vee \Diamond q)$	asp
3	$\mid\ \mid\ \neg\Box p \& \neg\Diamond q$	De Morgan's Theorem
4	$\mid\ \mid\ \neg\Box p$	conj elim
5	$\mid\ \mid\ \neg\Diamond q$	conj elim
6	$\mid\ \mid\ \Box\neg q$	def
7	$\mid\ \mid\ \mid\ \Box$	
8	$\mid\ \mid\ \mid\ p \vee q$	T-reit
9	$\mid\ \mid\ \mid\ \neg q$	T-reit
10	$\mid\ \mid\ \mid\ p$	dis elim
11	$\mid\ \mid\ \Box p$	nec intro
12	$\mid\ \mid\ \perp$	
13	$\mid\ \Box p \vee \Diamond q$	RAA
14	$\Box(p \vee q) \supset (\Box p \vee \Diamond q)$	impl intro

3. $\Box(p \vee q) \supset \Box(\Box p \vee \Diamond q)$

1	$\mid\ \Box(p \vee q)$	asp
2	$\mid\ \mid\ \Box$	
3	$\mid\ \mid\ \Box(p \vee q)$	S4-reit
4	$\mid\ \mid\ \mid\ \neg(\Box p \vee \Diamond q)$	asp
5	$\mid\ \mid\ \mid\ \neg\Box p \& \neg\Diamond q$	De Morgan's Theorem
6	$\mid\ \mid\ \mid\ \neg\Box p$	conj elim
7	$\mid\ \mid\ \mid\ \neg\Diamond q$	conj elim
8	$\mid\ \mid\ \mid\ \Box\neg q$	def
9	$\mid\ \mid\ \mid\ \mid\ \Box$	
10	$\mid\ \mid\ \mid\ \mid\ p \vee q$	T-reit
11	$\mid\ \mid\ \mid\ \mid\ \neg q$	T-reit
12	$\mid\ \mid\ \mid\ \mid\ p$	disj elim
13	$\mid\ \mid\ \mid\ \Box p$	nec intro
14	$\mid\ \mid\ \mid\ \bot$	
15	$\mid\ \mid\ \Box p \vee \Diamond q$	RAA
16	$\mid\ \Box(\Box p \vee \Diamond q)$	nec intro
17	$\Box(p \vee q) \supset \Box(\Box p \vee \Diamond q)$	impl intro

4. $\Box(p\&\Box q) \supset \Diamond p\&\Diamond q$

1	$\Box(p\&\Box q)$	asp
2	$\quad \Box$	
3	$\quad p\&\Box q$	T-reit
4	$\quad p$	conj elim
5	$\quad \Box q$	conj elim
6	$\quad q$	nec elim
7	$\Box p$	nec intro
8	$\Box q$	nec intro
9	p	nec elim
10	q	nec elim
11	$\Diamond p$	poss intro
12	$\Diamond q$	poss intro
13	$\Diamond p\&\Diamond q$	conj intro
14	$\Box(p\&\Box q) \supset \Diamond p\&\Diamond q$	

Abbreviations

AS4	axiom sysytem 4
AS5	axiom system 5
asp	assumption
conj elim	conjunction elimination
conj intro	conjunction introduction
def	definition
disj elim	disjunction elimination
disj intro	disjunction introduction
DN	double negation elimination
$\equiv$ elim	equivalence elimination
$\equiv$ intro	equivalence introduction
impl intro	implication introduction
MP	modus ponens (implication elimination)
MT	modus tollens
nec elim	necessity elimination
nec intro	necessity introduction
poss elim	possibility elimination
poss intro	possibility introduction
prem	premiss
RAA	reductio ad absurdum
reit	reiteration
rep	repetition
S4-reit	S4-reiteration
S5-reit	S5-reiteration
T-reit	T-reitalation